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RESEARCH MEMORANDUM

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RESEARCH MEMORANDUM

ON THE COMPUTATIONAL SOLUTION
OF SOME FUNCTIONAL EQUATIONS
IN THE THEORY OF DYNAMIC PROGRAMMING

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Summary: It is shown that functional equations of the form $f(p) = \operatorname{Max}_k T_k(f(p))$ which occur in the theory of optimal inventory and optimal allocation may be solved by very simple iterative processes under certain conditions.

ON THE COMPUTATIONAL SOLUTION OF SOME FUNCTIONAL EQUATIONS IN THE THEORY OF DYNAMIC PROGRAMMING

Richard Bellman

§1. Introduction

The theory of dynamic programming gives rise to a general class of functional equations having the form

$$f(p) = \operatorname{Max}_k T_k(f(p)) \tag{1.1}$$

where T_k is a transformation. In general, these equations cannot be resolved explicitly, and iterative techniques must be used to determine the solution.

Since the numerical evaluation of the maximum over k is, in general, quite onerous, it is desirable to have an alternative method of computation of successive approximation to the solutions.

We shall show how an alternative method can be obtained for two important classes of functional equations, those occurring in multi-stage allocation processes, an example of which is

$$f(x) = \operatorname{Max}_{0 \leq y \leq x} \left[g(y) + h(x-y) + f(ay + b(x-y)) \right] \tag{1.2}$$

and those occurring in optimal inventory theory, an example of which is

$$f(x) = \underset{y \geq x}{\mathrm{Min}} \left[v(x,y) + \lambda \left[f(0)(1-h(y)) + \int_0^y f(y-s)h'(s)ds \right] \right] \qquad (1.3)$$

§2. Optimal Allocation

It is known (see [1], [2]) that if

a. $0 < a, b < 1$

b. $g(0) = h(0) = 0$ (2.1)

c. $g(x)$ and $h(x)$ are continuous in $[0,c]$

then the unique continuous solution to (1.2) in $[0,c]$ which is zero at $x = 0$ is obtained by starting with an arbitrary continuous function $f_o(x)$ zero at $x = 0$, and iterating

$$f_1 = T(f_o)$$
$$f_2 = T(f_1) \qquad (2.2)$$

and so on, with $f(x) = \lim_{u \to \infty} f_n$.

Let us now assume that in (1.2) the maximum is taken inside the interval $[0,x]$ for all $x > 0$, and, indeed, for each approximation in (2.2) above.

Then we have

$$f_{n+1}(x) = g(y_n) + h(x-y_n) + f_n(ay_n + b(x-y_n)) \tag{2.3}$$

where $y_n = y_n(x)$ is determined by the equation

$$0 = g'(y) - h'(x-y) + (a-b)f_n'(ay + b(x-y)) \tag{2.4}$$

Returning to (2.3), we obtain

$$\begin{aligned} f_{n+1}'(x) &= [g'(y_n) - h'(x-y_n) + (a-b)f_n'(ay_n + b(x-y_n))] \frac{dy_n}{dx} \\ &\quad + h'(x-y_n) + bf_n'(ay_n + b(x-y_n)) \\ &= h'(x-y_n) + f_n'(ay_n + b(x-y_n)) \end{aligned} \tag{2.5}$$

From (2.4) we see that $f_0'(x)$ determines $y_0(x)$, while (2.5) shows that $y_0(x)$ and $f_0'(x)$ determine $f_1'(x)$, and so on, recurrently. From this it is clear that the important sequence is actually $\{f_n'(x)\}$, which may now be computed recurrently in very simple fashion.

Actually, we obain more. If $f_0(x)$ agrees with the actual solution $f(x)$ in some interval $[0,c]$, $f_1(x)$ will agree in a larger interval, $f_2(x)$ in a still larger interval, and so on. Thus the actual solution may be obtained by recurrence.

§3. Optimal Inventory

It is known (see [1], [4]) that if

a. $0 < \lambda < 1$

b. $h'(s) \geq 0, \int_0^\infty h'(s)ds = 1$ (3.1)

c. $v(x,y)$ is continuous in any finite interval $0 \leq y \leq c$,

the solution of (1.3) may be obtained by iteration using the sequence

$$f_{n+1} = T(f_n) \tag{3.2}$$

Let us assume again that the minimum is always taken inside. Then

$$f_{n+1}(x) = v(x,y_n) + \lambda\left[f_n(0)(1-h(y_n)) + \int_0^{y_n} f_n(y_n-s)h'(s)ds\right] \tag{3.3}$$

where y_n is determined by the equation

$$\frac{\partial v}{\partial y}(x,y_n) + \int_0^{y} f_n'(y-s)h'(s)ds = 0 \tag{3.4}$$

Returning to (3.3), we obtain

$$\begin{aligned} f_{n+1}'(x) &= \frac{\partial v}{\partial x}(x,y_n) \\ &\quad + \frac{dy_n}{dp}\left[g'(y_n-x) + \lambda\int_0^{y_n} f_n'(y_n-s)h'(s)ds\right] \\ &= \frac{\partial v}{\partial x}(x,y_n) \end{aligned} \tag{3.5}$$

This again furnishes a very simple computational algorithm for determining y_o given f_o' and then f_1' given y_o and f_o', and so on.

§4. How to Obtain Internal Extrema

To insure that the minimum in (3.2) is attained inside, we replace $g(y-x)$ by $g(y-x) - \varepsilon\sqrt{y-x}$ (or some similar function) where ε is small. Then the derivative at $y = x$ is $-\infty$, showing that the minimum cannot be at $y = x$.

Similarly in (2.2), we replace $g(y)$ by $g(y) + \varepsilon\sqrt{y}$ and $h(x-y)$ by $h(x-y) + \varepsilon\sqrt{x-y}$.

It is easily seen that for small ε these changes affect the solution by a small quantity proportional to ε, and consequently have no material influence.

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